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INFINITE-DIMENSIONAL WHITEHEAD AND VIETORIS THEOREMS IN SHAPE AND PRO-HOMOTOPY

BY

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ABSTRACT. In Theorem 3.3 and Remark 3.4 conditions are given under which an infinite-dimensional Whitehead theorem holds in pro-homotopy. Applications to shape theory are given in Theorems 1.1, 1.2, 4.1 and 4.2.

1. **Introduction.** Whitehead theorems in the shape theory of finite-dimensional spaces have been proved by Moszyńska [26] and by Mardešić [22], while in [7] we proved a Whitehead theorem in pro-homotopy theory for inverse systems of complexes whose dimensions are bounded. On first sight, the prospects for removing the hypotheses of finite dimension looked bleak, because of the counterexamples in [13], [11], [1, p. 35], [5] and [4]. However, by restricting ourselves just enough to avoid these counterexamples we have been able to prove reasonable theorems. We were led to them by reading the papers of Borsuk [31] and of Kozlowski and Segal [17]. For compact metric spaces (compacta) their theorem reads: a *movable compactum whose shape groups are trivial is shape equivalent to a point.* Our generalizations of this are Theorems 4.1 and 4.2 below. Confining ourselves in this introduction to the compact metric case, our theorem becomes:

THEOREM 1.1. *Let φ: $X \longrightarrow Y$ be a pointed shape morphism between pointed connected compacta, which induces isomorphisms on the (inverse limit) shape groups. If X is movable and Y is an FANR (in the pointed sense) then φ is a pointed shape equivalence.*

This is proved by combining Theorem 4.2, below, with Theorem 5.1 of our paper [7].

Theorem 1.1 has a geometrical consequence of some interest. A map f: $X \longrightarrow Y$ between compacta is called a *CE map* (or *cell-like map* or *Vietoris map*) if for each $y \in Y$, $f^{-1}(y)$ is shape equivalent to a point. If X and Y are ANR's then f must be a homotopy equivalence (see [12], [15] and the references

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therein). If X and Y are finite-dimensional compacta, f must be a shape equivalence (see [28] and [15]): Anderson (unpublished) was able to remove the requirement that X be finite dimensional. But if one also removes the requirement that Y be finite dimensional, counterexamples exist: Taylor [29] constructed a CE map from a nonmovable compactum onto the Hilbert Cube, while Keesling [14] modified this example to get a CE map from the Hilbert Cube onto a nonmovable compactum: clearly these cannot be shape equivalences. Kozlowski and Segal have gone further, by constructing [32] a CE map between movable compacta of different shapes. The theorems in this paper imply the following "infinite-dimensional Vietoris theorem" (which is proved by combining Theorem 1.1, above, with Theorem 2.3 of K. Kuperberg's paper [18]).

THEOREM 1.2. *Let* $f\colon (X, x) \longrightarrow (Y, y)$ *be a CE map between pointed connected compacta. If* (X, x) *is movable and* (Y, y) *is an FANR* (*in the pointed sense*), *then* f *is a pointed shape equivalence.*

Our principal tool is a Whitehead theorem in pro-homotopy, Theorem 3.3. Roughly, it says that a weak equivalence in pro-homotopy from an inverse system $\{X_\alpha\}$ of finite-dimensional complexes to a finite-dimensional complex Y is an equivalence provided $\{X_\alpha\}$ is movable. The point is that the dimensions of the complexes X_α need not be bounded.

In Remarks 3.4 and 4.4 we indicate how the hypotheses on X and Y of Theorem 1.1 can be replaced by the hypothesis that φ be a "movable morphism."

NOTE ADDED MAY 1, 1975. J. Dydak [39] has extended our shape theoretic results. It is not clear whether his methods can be adapted to improve our pro-homotopy results.

2. Notation, terminology and a lemma. We follow the notational conventions set out in §§2 and 3 of [7]. The principal items are listed below. Shape terminology is introduced in §4.

If C is a category, pro-C denotes a certain category of inverse systems in C indexed by directed sets: for a description of the morphisms and other properties of pro-C see [5] or [22]; for the original more general version see the Appendix of [1]. C_{maps} denotes the category whose objects are the morphisms of C and whose morphisms from an object f to an object g are the commutative square diagrams

$$\begin{array}{ccc} \cdot & \xrightarrow{f} & \cdot \\ \downarrow & & \downarrow \\ \cdot & \xrightarrow[g]{} & \cdot \end{array}$$

in C. There is an obvious functor pro-$(C_{\mathrm{maps}}) \longrightarrow (\text{pro-}C)_{\mathrm{maps}}$ and we say that the object $\{X_\alpha \xrightarrow{f_\alpha} Y_\alpha\}$ of pro-(C_{maps}) "induces" its image $\{X_\alpha\} \xrightarrow{\{f_\alpha\}} \{Y_\alpha\}$ under this functor: see §3 of [7].

We suppress bonding morphisms and the indexing directed set, denoting an object of pro-C by $\{X_\alpha\}$. $\{X_\alpha\}$ is *movable* if for each α there exists $\beta \geqslant \alpha$ such that for all $\gamma \geqslant \beta$ the bond $p_{\alpha\beta}$: $X_\beta \longrightarrow X_\alpha$ factors as $p_{\alpha\beta} = p_{\alpha\gamma} \circ r^{\beta\gamma}$ where $r^{\beta\gamma}$: $X_\beta \longrightarrow X_\gamma$ is a morphism of C. $\{X_\alpha\}$ is *uniformly movable* if for each α there exists $\beta \geqslant \alpha$ such that the bond $p_{\alpha\beta}$ factors as $p_{\alpha\beta} = p_\alpha \circ r^\beta$ where r^β: $X_\beta \longrightarrow \{X_\alpha\}$ is a morphism of pro-C and p_α: $\{X_\alpha\} \longrightarrow X_\alpha$ is the projection morphism of pro-C. (C is, of course, embedded as a full subcategory of pro-C.)

A directed set is *closure finite* if each element has only finitely many predecessors.

Categories used include: Groups (groups and homorphisms); T_0 (pointed connected topological spaces and pointed continuous functions); HT_0 (the homotopy category corresponding to T_0); CW_0 (pointed connected CW complexes and pointed continuous functions); H_0 (the homotopy category corresponding to CW_0); $HT_{0,\mathrm{pairs}}$ (pointed pairs of connected spaces and pointed homotopy classes of maps); $H_{0,\mathrm{pairs}}$ (pointed pairs of connected CW complexes and pointed homotopy classes of maps).

We call an object of pro-Groups a *pro-group.* We always suppress base points of spaces.

The definition of uniform movability becomes simpler in the case of pro-groups. A pro-group $G \equiv \{G_\alpha\}$ is (clearly) uniformly movable if and only if for each α there exists $\beta \geqslant \alpha$ such that the bond $p_{\alpha\beta}$: $G_\beta \longrightarrow G_\alpha$ factors as $p_{\alpha\beta} = p_\alpha \circ r^\beta$ where r^β: $G_\beta \longrightarrow \varprojlim G$ is a homomorphism and p_α: $\varprojlim G \longrightarrow G_\alpha$ is projection on the α factor.

LEMMA 2.1. *Let $G \equiv \{G_\alpha\}$ be a uniformly movable pro-group. Let H be a group, let f: $G \longrightarrow H$ be a morphism of* pro-Groups *and let p: $\varprojlim G \longrightarrow G$ be the projection morphism. If $\widetilde{f} \equiv f \circ p$ is an isomorphism* (*of groups*) *then f is an isomorphism* (*of* pro-groups).

PROOF. The required inverse to f is $p \circ \widetilde{f}^{-1}$. It is trivial that $f \circ (p \circ \widetilde{f}^{-1})$ is the identity of G. To show that $(p \circ \widetilde{f}^{-1}) \circ f$ is the identity of $\{G_\alpha\}$ it is enough to show that given α there exists $\gamma \geqslant \alpha$ such that $p_\alpha \circ \widetilde{f}^{-1} \circ f_\beta \circ p_{\beta\gamma} = p_{\alpha\gamma}$. Since G is uniformly movable the above remark implies that there exist $\beta \geqslant \alpha$ and r^β: $G_\beta \longrightarrow \varprojlim G$ such that $p_{\alpha\beta} = p_\alpha \circ r^\beta$. Let f be represented by homomorphisms f_α: $G_\alpha \longrightarrow H$. Choose $\gamma \geqslant \beta$ such that $f_\beta \circ p_{\beta\gamma} = f_\alpha \circ p_{\alpha\beta} \circ p_{\beta\gamma}$. Then

$$f_\beta \circ p_{\beta\gamma} = f_\alpha \circ p_\alpha \circ r^\beta \circ p_{\beta\gamma} = \widetilde{f} \circ r^\beta \circ p_{\beta\gamma}.$$

So

$$p_\alpha \circ \widetilde{f}^{-1} \circ f_\beta \circ p_{\beta\gamma} = p_\alpha \circ r^\beta \circ p_{\beta\gamma} = p_{\alpha\beta} \circ p_{\beta\gamma} = p_{\alpha\gamma}. \qquad \square$$

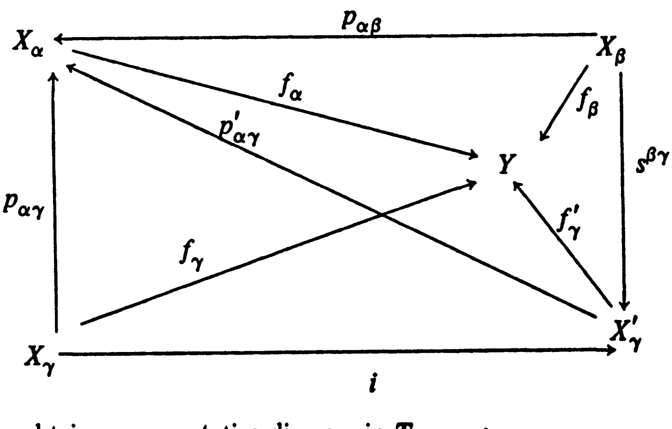
3. A Whitehead theorem in pro-homotopy. The principal result here is Theorem 3.3. Lemma 3.1 and Proposition 3.2 are the modifications needed to obtain an infinite-dimensional Whitehead theorem from [22] and [7].

If $f: X \longrightarrow Y$ is a morphism of T_0, $M(f)$ denotes the reduced mapping cylinder of f, and X is regarded as a subset of $M(f)$ in the usual manner. Thus $(M(f), X)$ is an object of $T_{0,\mathrm{pairs}}$. If $f \equiv \{X_\alpha \xrightarrow{f_\alpha} Y_\alpha\}$ is an object of pro-$(T_{0,\mathrm{maps}})$ then $M(f) \equiv \{(M(f_\alpha), X_\alpha)\}$ is a well-defined object of pro-$(T_{0,\mathrm{pairs}})$ and so induces an object of pro-$(HT_{0,\mathrm{pairs}})$; see §3 of [7].

LEMMA 3.1. *Let $f \equiv \{X_\alpha \xrightarrow{f_\alpha} Y_\alpha\}$ be an object of* pro-$(CW_{0,\mathrm{maps}})$ *whose domain $\{X_\alpha\}$ is movable in* pro-H_0 *and whose range $\{Y_\alpha\}$ is such that every Y_α is the same (pointed) complex Y, and every bond $Y_\beta \longrightarrow Y_\alpha$ is the identity map. Then $\{(M(f_\alpha), X_\alpha)\}$ is movable in* pro-$(HT_{0,\mathrm{pairs}})$.

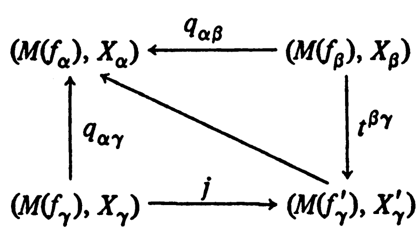
PROOF. Let $p_{\alpha\beta}: X_\beta \longrightarrow X_\alpha$ denote the appropriate bond of $\{X_\alpha\}$. Given α, there exists $\beta \geqslant \alpha$ such that for all $\gamma \geqslant \beta$ there is a pointed map $r^{\beta\gamma}: X_\beta \longrightarrow X_\gamma$ with the property that $p_{\alpha\beta}$ is pointedly homotopic to $p_{\alpha\gamma} \circ r^{\beta\gamma}$. By Theorem 2.8.9 of [37], $p_{\alpha\gamma}$ may be replaced by a fibration: to be precise, there exist a pointed fibration $p'_{\alpha\gamma}: X'_\gamma \longrightarrow X_\alpha$, and an inclusion $i: X_\gamma \longrightarrow X'_\gamma$ as a pointed strong deformation retract, such that $p'_{\alpha\gamma} \circ i = p_{\alpha\gamma}$. Since $p'_{\alpha\gamma}$ is a fibration, there is a pointed map $s^{\beta\gamma}: X_\beta \longrightarrow X'_\gamma$ such that $p'_{\alpha\gamma} \circ s^{\beta\gamma} = p_{\alpha\beta}$. Letting $f'_\gamma = f_\alpha \circ p'_{\alpha\gamma}$ we have a commutative diagram in CW_0:

(*)



From it, we obtain a commutative diagram in $T_{0,\mathrm{pairs}}$:

(**)



$\bar{j}$ induces pointed homotopy equivalences $M(f_\gamma) \longrightarrow M(f'_\gamma)$ and $X_\gamma \longrightarrow X'_\gamma$; by Lemma 1 of [34], we could deduce that $\bar{j}$ induces an isomorphism in $HT_{0,\text{pairs}}$ if we knew that $(M(f_\gamma), X_\gamma)$ and $(M(f'_\gamma), X'_\gamma)$ were isomorphic in $HT_{0,\text{pairs}}$ to pointed CW pairs. It is not hard to show that $(M(f_\gamma), X_\gamma)$ has this property (f_γ is homotopic to a cellular map; use Lemma 3.9 of [7]). But it is not clear that the same is true of $(M(f'_\gamma), X'_\gamma)$. To avoid the problem, we apply the composite functor "geometric realization of the singular complex" [33], $|\cdot| \circ S$: $T_0 \longrightarrow CW_0$, to the diagram (*), and thus obtain the following commutative diagram in $CW_{0,\text{pairs}}$, analogous to (**):

$$\begin{array}{ccc}
(M(|Sf_\alpha|), |SX_\alpha|) & \xleftarrow{\ \bar{q}_{\alpha\beta}\ } & (M(|Sf_\beta|), |SX_\beta|) \\
\uparrow{\scriptstyle \bar{q}_{\alpha\gamma}} & \nwarrow & \downarrow{\scriptstyle \bar{t}^{\beta\gamma}} \\
(M(|Sf_\gamma|), |SX_\gamma|) & \xrightarrow{\ \bar{j}\ } & (M(|Sf'_\gamma|), |SX'_\gamma|)
\end{array}$$

where the maps are obtained from those of (**) in the obvious way. Now, Lemma 1 of [34] implies that $\bar{j}$ induces an isomorphism in $HT_{0,\text{pairs}}$. It follows that $\bar{q}_{\alpha\beta}$ can be lifted in $HT_{0,\text{pairs}}$ through $\bar{q}_{\alpha\gamma}$, so that $\{(M(|Sf_\alpha|), |SX_\alpha|)\}$ is movable in $HT_{0,\text{pairs}}$ (where the bonds are induced by the maps $\bar{q}_{\alpha\beta}$). The argument is completed by observing that $\{(M(|Sf_\alpha|), |SX_\alpha|)\}$ is isomorphic in pro-$HT_{0,\text{pairs}}$ to $\{(M(f_\alpha), X_\alpha)\}$. To see this, observe that there is a commutative diagram in $T_{0,\text{pairs}}$

$$\begin{array}{ccc}
(M(f_\alpha), X_\alpha) & \longleftarrow & (M(|Sf_\alpha|), |SX_\alpha|) \\
\uparrow{\scriptstyle q_{\alpha\beta}} & & \uparrow{\scriptstyle \bar{q}_{\alpha\beta}} \\
(M(f_\beta), X_\beta) & \longleftarrow & (M(|Sf_\beta|), |SX_\beta|)
\end{array}$$

whose horizontal morphisms are induced by the canonical maps $|SX_\alpha| \longrightarrow X_\alpha$, $|SY| \longrightarrow Y$, etc. As explained above $(M(f_\alpha), X_\alpha)$ is isomorphic in $HT_{0,\text{pairs}}$ to a pointed CW pair; and $(M(|Sf_\alpha|), |SX_\alpha|)$ is itself a pointed CW pair. So, by Lemma 1 of [34], the horizontal morphisms are isomorphisms in $HT_{0,\text{pairs}}$. Since $\{M(f_\alpha)\}$ is isomorphic to a movable object, it is itself movable. □

PROPOSITION 3.2. *Let $\{(P_\alpha, P'_\alpha)\}$ be a movable object of* pro-$(H_{0,\text{pairs}})$ *indexed by a closure finite directed set. Assume that each P_α is a finite-dimensional simplicial complex and that P'_α is a subcomplex of P_α. If $\{\pi_k(P_\alpha, P'_\alpha)\}$ is trivial for all k, then the "inclusion" $\{P'_\alpha\} \longrightarrow \{P_\alpha\}$ is an isomorphism in* pro-H_0.

PROOF. The proof is almost identical to that of Theorem 2 of [22]. Movability makes unnecessary the hypothesis in [22] that the dimensions of the complexes P_α be bounded. For each α choose $\beta(\alpha) \geqslant \alpha$ such that for every $\gamma \geqslant \beta(\alpha)$ there exists a morphism of $H_{0,\text{pairs}}$, $s^{\beta\gamma}$: $(P_\beta, P'_\beta) \longrightarrow (P_\gamma, P'_\gamma)$, such that $q_{\alpha\gamma} \circ s^{\beta\gamma} = q_{\alpha\beta}$ where $q_{\lambda\mu}$: $(P_\lambda, P'_\lambda) \longrightarrow (P_\mu, P'_\mu)$ denotes the appropriate bonding morphism. Following 2.3 of [22], assume $\beta(\alpha) \leqslant \beta(\bar{\alpha})$ whenever $\alpha \leqslant \bar{\alpha}$.

Claim 1. For each α, each pointed pair of finite-dimensional simplicial complexes (K, K') and each pointed map φ: $(K, K') \longrightarrow (P_{\beta(\alpha)}, P'_{\beta(\alpha)})$ there exists a pointed map ψ: $K \longrightarrow P'_\alpha$ such that (inclusion) $\circ\ \psi$ is pointedly homotopic to (bond) $\circ\ \varphi$ in P_α and $\psi|K'$ is pointedly homotopic to (bond) $\circ\ \varphi|K'$ in P'_α.

PROOF OF CLAIM 1. Apply Lemma 1 (§6.2) of [22] with $n + 1 = \dim K$ and $\alpha^* \geqslant \beta(\alpha)$: movability implies that φ can be lifted to $(P_{\alpha^*}, P'_{\alpha^*})$, hence ψ exists.

Claim 2. Given α and a pointed finite-dimensional complex L, let φ_0, φ_1: $L \longrightarrow P'_{\beta(\alpha)}$ be pointed maps such that (inclusion) $\circ\ \varphi_0$ and (inclusion) $\circ\ \varphi_1$ are pointedly homotopic in $P_{\beta(\alpha)}$. Then (bond) $\circ\ \varphi_0$ and (bond) $\circ\ \varphi_1$ are pointedly homotopic in P'_α.

PROOF OF CLAIM 2. Apply Lemma 2 (§6.3) of [22] with $n = \dim L$: movability implies that φ_0 and φ_1 can be lifted to P'_{α^*}, and the claim follows.

The remainder of the proof is similar to the corresponding proof in §6.4 of [22]. Claims 1 and 2 are used in place of Lemmas 1 and 2 of [22]. □

THEOREM 3.3. *Let Y be a pointed complex, $\{X_\alpha\}$ an object of* pro-CW_0 *and g: $X \longrightarrow Y$ a morphism of* pro-H_0. *Assume Y and each X_α are finite dimensional, and that the object of* pro-H_0 *induced by $\{X_\alpha\}$ is movable. If $g_\#$: $\{\pi_k(X_\alpha)\} \longrightarrow \pi_k(Y)$ is an isomorphism (in the category* pro-Groups) *for every k, then g induces an isomorphism of* pro-H_0.

PROOF. The proof is similar to that of Theorem 3.1 of [7]. Since Y is a complex, we may represent g by a morphism of pro-CW_0 and hence replace it by an object $f \equiv \{X'_\gamma \xrightarrow{f_\gamma} Y'_\gamma\}$ of pro-$(CW_{0,\text{maps}})$ indexed by a closure finite directed set such that: $\{X'_\gamma\}$ is movable, each X'_γ is finite dimensional, each Y'_γ is Y, and each bond of $\{Y'_\gamma\}$ is the identity map; see §3 of [7]. $f_\#$: $\{\pi_k(X'_\gamma)\} \longrightarrow \{\pi_k(Y'_\gamma)\}$ is an isomorphism of pro-groups for each k. By Lemma 3.8 of [7], $\{\pi_k(M(f_\gamma), X'_\gamma)\}$ is trivial, where $\{M(f_\gamma)\}$ is the reduced mapping cylinder object of pro-CW_0 corresponding to f (see §3 of [7]). Each $M(f_\gamma)$ is finite dimensional. By Lemma 3.1, above, $\{(M(f_\gamma), X'_\gamma)\}$ is movable in pro-$(HT_{0,\text{pairs}})$. The rest of the proof is as in [7], except that Proposition 3.12 of [7] is replaced by the above Proposition 3.2. □

REMARK 3.4. There is a variation on Theorem 3.3. Following [9], define H-$CW_{0,\text{maps}}$ to be the category whose objects are those of $CW_{0,\text{maps}}$ and whose morphisms are homotopy classes of morphisms of $CW_{0,\mathrm{maps}}$, where two morphisms (a_1, a_2) and (b_1, b_2) from $f: X \longrightarrow Y$ to $f': X' \longrightarrow Y'$ are defined to be *homotopic* if there is a morphism (θ_1, θ_2) from $f \times 1: X \times I \longrightarrow Y \times I$ to $f': X' \longrightarrow Y'$ such that θ_i is a homotopy between a_i and b_i, $i = 1, 2$. Call an object $\{X'_\gamma \xrightarrow{f_\gamma} Y'_\gamma\}$ of pro-$CW_{0,\mathrm{maps}}$ *H-movable* if it induces a movable object of pro-$(H\text{-}CW_{0,\mathrm{maps}})$. Call a morphism $g: \{X_\alpha\} \longrightarrow \{Y_\beta\}$ of pro-CW_0 *movable* if g is isomorphic in $(\text{pro-}H_0)_{\mathrm{maps}}$ to the object of $(\text{pro-}H_0)_{\mathrm{maps}}$ induced by such an H-movable $\{f_\gamma\}$. *If each X_α and each Y_β is finite dimensional, if g is movable, and if g induces isomorphisms of homotopy pro-groups, then g induces an isomorphism in* pro-H_0. The proof is similar to that of Theorem 3.3. The hypotheses make it possible to by-pass Lemma 3.1: clearly $\{(M(f_\gamma), X'_\gamma)\}$ is movable in pro-$(HT_{0,\mathrm{pairs}})$.

4. **Whitehead theorems in shape.** All spaces mentioned will be paracompact Hausdorff, so our shape theory may be understood either in the sense of [21] or [27], since these two theories agree on such spaces [19], [25]. For compact Hausdorff spaces these theories agree with that of [23], and for compact metric spaces they agree with that of [3] (see [24]).

We refer the reader to [25] or to §3 of [22] for an account of how the shape theory of spaces is fully and faithfully reflected in pro-homotopy theory. In particular, if X and Y are pointed connected spaces, there is a functorial bijection between the (pointed) shape morphisms from X to Y and the morphisms of pro-H_0 from $\{X_\alpha\}$ to $\{Y_\beta\}$, where $\{X_\alpha\}$ and $\{Y_\beta\}$ are objects of pro-H_0 (unique up to isomorphism) which are "associated" with X and Y respectively. A shape morphism $\varphi: X \longrightarrow Y$ is a *weak shape equivalence* if the corresponding $f: \{X_\alpha\} \longrightarrow \{Y_\beta\}$ induces isomorphisms $f_\#: \{\pi_k(X_\alpha)\} \longrightarrow \{\pi_k(Y_\beta)\}$ in pro-Groups for each $k \geqslant 1$. φ is a *very weak shape equivalence* if

$$f_\#: \varprojlim_\alpha \{\pi_k(X_\alpha)\} \longrightarrow \varprojlim_\beta \{\pi_k(Y_\beta)\}$$

is an isomorphism in Groups for each $k \geqslant 1$. X is *movable* [resp. *uniformly movable*] if $\{X_\alpha\}$ is movable [resp. uniformly movable] in pro-H_0.

Every object of pro-CW_0 gives rise to an object of pro-H_0, but (apart from the case of countably indexed systems) it is unknown whether every object of pro-H_0 "comes from" an object of pro-CW_0. The Vietoris functor [27] allows one to associate objects "coming from" pro-CW_0 with spaces, but the complexes involved are infinite dimensional. It is for these reasons that we confine ourselves to compact Hausdorff spaces in the theorems which follow.

THEOREM 4.1. *Let X be a movable pointed connected compact Hausdorff space, let Y be pointed shape equivalent to a pointed connected CW complex* *and let φ: $X \longrightarrow Y$ be a pointed shape morphism. If φ is a weak shape equivalence, it is a pointed shape equivalence.*

PROOF. Assume Y is a CW complex. First assume Y is a finite-dimensional complex. As we shall see, no generality is lost by this.

Let $\{X_\alpha\}$ be an object of pro-CW_0 whose inverse limit is homeomorphic to X. Then $\{X_\alpha\}$ is associated with X in the sense of [25]. Let g: $\{X_\alpha\} \longrightarrow Y$ be a morphism of pro-H_0 associated with φ in the sense of [25]. By Theorem 3.3, g induces an isomorphism in pro-H_0. Hence, by [25], φ is a shape equivalence.

If Y is not finite dimensional we show that it must be (pointed) homotopy equivalent to a finite-dimensional complex. Since X is compact, g may be represented by a continuous map g_{α_0}: $X_{\alpha_0} \longrightarrow Y$ for some α_0, and hence g factors through a finite subcomplex K of Y. So $\widetilde{g}$: $\{\widetilde{X}_\alpha\} \longrightarrow \widetilde{Y}$ factors through $\widetilde{K}$ (where we have applied the pointed universal cover functor $\widetilde{\ }$). Since g is a weak equivalence in pro-H_0, so also is $\widetilde{g}$. Hence g and $\widetilde{g}$ are $\natural$-isomorphisms [1, §4]; therefore, they induce isomorphisms on homology pro-groups and cohomology groups with every possible coefficient bundle (see 4.4 of [1]). Since K and $\widetilde{K}$ are finite dimensional, the homology of $\widetilde{Y}$ and the cohomology of Y vanish above the dimension of K. By Theorem E of [30], Y is homotopy equivalent (hence pointed homotopy equivalent) to a finite-dimensional complex. □

THEOREM 4.2. *Let X be a uniformly movable pointed connected compact Hausdorff space, let Y be pointed shape equivalent to a pointed CW complex, and let φ: $X \longrightarrow Y$ be a morphism in pointed shape. If φ is a very weak shape equivalence, it is a pointed shape equivalence. Furthermore, if X is metrizable it is only necessary to assume that X is movable.*

PROOF. By Lemma 2.1, φ is a weak shape equivalence, so the conclusion follows from Theorem 4.1. For metric compacta the concepts of "movable" and "uniformly movable" coincide, by [38] (see also Theorem 4.7 of [16] and Remark 6.7 of [35]) so the last statement is justified.

REMARK 4.3. Various criteria are available for deciding if a given space Y is shape equivalent to a CW complex (as required in Theorems 4.1 and 4.2). See [10], [6], [7], [8].

REMARK 4.4. Following Remark 3.4, one may define the notion of "movable shape morphism": the special case of "movable map" is discussed in [9]. One may then prove that *if φ: $X \longrightarrow Y$ is a movable pointed shape morphism between metric compacta and if φ is a very weak shape equivalence, then φ is a shape equivalence.* A remark on p. 4 of [2] (incorrect as stated, but correct in the countable case) is used instead of Lemma 2.1 to show that φ is a weak shape equivalence. Then Remark 3.4 is used instead of Theorem 3.3 to complete the proof. Compare with [36].

REMARK 4.5. If one interchanges the properties of X and Y in Theorems 4.1 and 4.2, making Y movable (or uniformly movable) and X shape equivalent to a complex, the resulting "theorems" are false. Counterexamples are given in [5]. However, if one also requires X to be compact metric (or, equivalently, to be an FANR: see [6]) we do not know a counterexample. Added in proof: there is none; see [39].

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